along the 40th parallel in North America between the Atlantic Ocean and Salt Lake City. Magnetic observations have been made, with more or less assiduity, at different places in the eastern States for many years past; but of the immense tract of country lying between the Mississippi and the Pacific Ocean there is only a single determination of one of the three elements indicated on Sir Edward Sabine's maps, viz., a determination of declination at Salt Lake City. A series of observations was made some years since by United States' officers along the Mexican frontier, and a similar series was carried out by the English and American officers employed on the North American Boundary Commission. The present set of observations was made, therefore, along the district which lies midway between the line of observations already run along the northern and southern boundaries of the United States' territory.

The instruments employed formed part of the admirable magnetic equipment belonging to Owens College, Manchester, and were kindly placed at the disposal of the author by the Council of the College. At the commencement and end of the survey a complete series of observations was made at Kew, which was adopted as a base station; and a similar set was made at Washington in the magnetic observatory belonging to Mr. Charles A. Schott, of the United States' Coast Survey Department, with a view of obtaining an independent check on the indications of the instruments. The author learned from Mr. Schott that the Coast Survey Department is gradually accumulating data for a discussion of the magnetic history of the American continent. It has already published a map of declination for 1870, from which it is evident that much has been done in the determination of this particular element since the date of Sir E. Sabine's memoirs, more especially along the Pacific slope, and in the regions to the west and south-west of the Great Salt Lake. As yet, however, no observations have been published relating to any of the districts west of the Mississippi visited by the author.

II. "On certain Definite Integrals occurring in Spherical Harmonic Analysis, and on the Expansion in Series of the Potentials of the Ellipsoid and of the Ellipse." By W. D. NIVEN, M.A., Fellow of Trinity College, Cambridge. Communicated by J. W. L. GLAISHER, M.A., F.R.S. Received April 3, 1879.

(Abstract.)

1. The object of this paper is to explain a general method of calculating a class of integrals connected with the expansion of functions in spherical harmonic series.

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The basis of the method is expressed by the evaluation of the integral

$$\iint e^{\alpha x+\beta y+\gamma z}\,d\mathrm{S}$$

taken over the surface of a sphere of radius R, whose centre is the origin of co-ordinates. This is easily shown to be

$$4\pi\mathrm{R}^2\left\{1+\frac{\mathrm{R}^2}{3\,!}(\alpha^2+\beta^2+\gamma^2)+\,..\,+\frac{\mathrm{R}^{2i}}{2i+1\,!}(\alpha^2+\beta^2+\gamma^2)^i+\,\ldots\right\}.$$

Hence it follows that, if V be any function which can be expressed symbolically in the form

$$e^{x\frac{d}{dx}+y\frac{d}{dy}+z\frac{d}{dz}}\mathrm{V}_0,$$

where the differential coefficients of V belong to the origin, and (x, y, z) is any point within the sphere, then

$$\iint \mathrm{V}\,d\mathrm{S}=4\pi\mathrm{R}^2\sum_0^\infty\frac{\mathrm{R}^{2i}}{2i+1\,!}\left(\frac{d^2}{dx}+\frac{d^2}{dy}+\frac{d^2}{dz}\right)^i\mathrm{V}_0\,\ldots\ldots\ldots\ldots\,(1).$$

If we next consider a circle in the plane of x, y, whose centre is the origin, and whose radius is R, we find, in like manner,

$$\int \mathrm{V}\,ds$$

taken round the perimeter, equal to

$$2\pi\mathrm{R}\sum_0^\infty\frac{\mathrm{R}^{2i}}{2^{2i}i\,!\,i\,!}\left(\frac{d^2}{dx}+\frac{d^2}{dy}\right)^i\mathrm{V}_0\,\ldots\ldots\ldots\ldots\ldots\ldots\,(2).$$

The theorems (1) and (2) are applied in a variety of cases, of which we will quote two results:—

Let $\mathrm{P}_n\ (\mu)$ be the zonal harmonic of the nth degree, having its pole in the axis of z, and let m and n be integers, then

$$(1)\quad \iint(1-\mu^2)^m\mathrm{P}_{2n}\,d\mathrm{S}=(-1)^n\frac{4^m}{2m+2n+1\,!}\;\frac{m+n\,!\;m\,!\;m\,!\;2n\,!}{m-n\,!\;n\,!\;n\,!}4\pi\mathrm{R}^2,$$

provided m be not less than n. In all other cases the integral vanishes, the cases where the harmonic is of odd degree included.

$$(2)\quad \iint(1-\mu^2)^{m-\frac{1}{2}}\mathrm{P}_{2n}\,d\mathrm{S}=$$

$$(\alpha)\quad n>m,\ (-1)^m\frac{2m\,!\;2n\,!\;2n\,!\;2n-2m\,!}{2^{2m+4n}n+m\,!\;n-m\,!\;m\,!\;m\,!\;n\,!\;n\,!}2\pi^2\mathrm{R}^2.$$

$$(\beta)\quad n<m,\ (-1)^n\frac{2m\,!\;2m\,!\;2n\,!\;m-n\,!}{2^{2m+4n}m+n\,!\;2m-2n\,!\;m\,!\;m\,!\;n\,!\;n\,!}2\pi^2\mathrm{R}^2.$$

In the "Proceedings of the Royal Society," vol. xxvii, pp. 63—71, Professor J. C. Adams has found the value of the integral

$$\iint P_p P_q P_r dS.$$

Mr. Ferrers had also independently succeeded in the evaluation, and had published the result in his "Treatise on Spherical Harmonics." In the paper of which this is an abstract, the same integral is discussed, and an attempt is made to extend the result to the cases of three tesseral harmonics, and of four zonals with coincident poles, but with only partial success.

2. If we seek to apply the same methods to ellipsoids and ellipses, we find that

$$\iint V p dS,$$

taken over the surface of the ellipsoid of semi-axes a, b, c, where p is the perpendicular from the centre upon dS, is equal to

$$4\pi abc\sum_0^\infty \frac{\left(a^2\frac{d^2}{dx}+b^2\frac{d^2}{dy}+c^2\frac{d^2}{dz}\right)^i}{2i+1!}V_0,$$

and that

$$\iiint V dx\, dy\, dz = 4\pi abc\sum_0^\infty \frac{1}{2i+3}\,\frac{\left(a^2\frac{d^2}{dx}+b^2\frac{d^2}{dy}+c^2\frac{d^2}{dz}\right)^i}{2i+1!}V_0.$$

As a particular case, let us consider V as the potential due to unit matter at the point (f, g, h) outside of the ellipsoid. Then, since

$$PQ^2 = (f-x)^2 + (g-y)^2 + (h-z)^2,$$

and therefore

$$\frac{d}{dx}\frac{1}{PQ} = -\frac{d}{df}\frac{1}{PQ},$$

$$\frac{d^2}{dx}\frac{1}{PQ} = \frac{d^2}{df}\frac{1}{PQ}, \text{ \&c.,}$$

it follows that we may take x, y, z, zero before differentiation, and write the two results :—

$$\iint \frac{p\,dS}{PQ} = 4\pi abc\sum_0^\infty \frac{\left(a^2\frac{d^2}{df}+b^2\frac{d^2}{dg}+c^2\frac{d^2}{dh}\right)^i}{2i+1!}\,\frac{1}{\sqrt{f^2+g^2+h^2}}, \quad \ldots\ldots\ldots (3)$$

a relation giving the potential of an ordinary ellipsoidal shell in spherical harmonics;

$$\iiint\frac{dx\,dy\,dz}{\mathrm{PQ}}=\mathrm{M}\sum_{0}^{\infty}\frac{3}{2i+3}\,\frac{\left(a^2\frac{d^2}{df}+b^2\frac{d^2}{dg}+c^2\frac{d^2}{dh}\right)^i}{2i+1\,!}\,\frac{1}{\sqrt{f^2+g^2+h^2}},\;\ldots\;(4)$$

where M is the mass of the ellipsoid. Equation (4) gives the harmonic expansion of a solid ellipsoid of unit density; and it will be observed that, in that form, it gives a very simple proof of Maclaurin's theorem, since the operator

$$a^2\frac{d^2}{df}+b^2\frac{d^2}{dg}+c^2\frac{d^2}{dh}$$

is unaffected by the addition of the terms

$$k\left(\frac{d^2}{df}+\frac{d^2}{dg}+\frac{d^2}{dh}\right).$$

By means of (4) it is easy to find what Thomson and Tait have called the exhaustion of potential energy, due to the mutual action of two solid ellipsoids, and then by spherical harmonic analysis, to determine approximately the nature of the forces acting between them.

Pursuing the same method, we next find that the potential at an outside point (f, g, h) due to an ellipse of density 1, with its axes $2a$, $2b$, in the axes of co-ordinates x, y, is

$$2\pi ab\sum_{0}^{\infty}\frac{\left(a^2\frac{d^2}{df}+b^2\frac{d^2}{dg}\right)^i}{2^{2i+1}\,i!\,i_{+1}!}\,\frac{1}{\sqrt{f^2+g^2+h^2}}\;\ldots\ldots\ldots\ldots\;(5).$$

From (5) it is easy to pass to the potential of a magnetic pole at f, g, h, due to unit current circulating in the boundary of the ellipse, and thence to the expansion in harmonics for the mutual potential energy of the currents in two elliptic circuits in any positions.

Results similar to (4) and (5) can also be easily found for solid parallelopipeds and for rectangular circuits.

The series here found cease to be convergent within certain regions including the origin. The expansion of the potential in those regions must be found by an independent investigation.

It may be remarked that the harmonics in the form of differential coefficients, given in (3), (4), and (5), may easily be expressed in terms of the corresponding surface harmonics, according to the following theorem :—

Let r be the radius vector from the origin O to any point P, and let $f\left(\frac{d}{dx},\frac{d}{dy},\frac{d}{dz}\right)$ be any homogeneous operator of the ith degree operating upon the reciprocal of r, then

$$f\left(\frac{d}{dx},\frac{d}{dy},\frac{d}{dz}\right)\frac{1}{r}=(-1)^i\frac{1}{r^{i+1}}f\left(\frac{d}{dx},\frac{d}{dy},\frac{d}{dz}\right)(\mathrm{Q}_i r^i),$$

where Q_i is the biaxal harmonic of the *i*th degree, whose pole is in OP, and which may therefore be expressed in the usual way, according to Laplace's addition formula. The differentiations are then easily performed.

III. "On the Origin of the Parallel Roads of Lochaber, and their bearing on other Phenomena of the Glacial Period." By JOSEPH PRESTWICH, M.A., F.R.S., F.G.S., &c., Professor of Geology in the University of Oxford. Received March 27, 1879.

(Abstract.)

Of the various hypotheses that have been brought forward since the time of Macculloch and Dick-Lauder to account for the origin of the Parallel Roads of Glen Roy, the one so ably propounded by Mr. Jamieson, in 1863, has been most generally received and adopted. It is a modification of the views originally expressed by Agassiz, to the effect that the barriers of the lakes,—to the shore action of which both the above-named geologists attributed the "roads," but were at a loss to account both for the formation and removal of barriers,—had been formed during the glacial period by glaciers issuing from Glen Treig and Glen Arkaig, supplemented by others from Ben Nevis. The subsequent determination, by the Scotch geologists, of an intermediate milder period succeeded by a second cold period, led Mr. Jamieson, with whom the preglacial and glacial deposits of Scotland had been a subject of especial investigation, to conclude that the extension of these two glaciers took place during the second cold period, which he thinks was of little less intensity than the first, and that, while the glacier from Glen Arkaig blocked up Glen Gluoy, the glacier from Glen Treig formed a barrier to Glen Roy. He observes, "Grant, then, these two ice-streams, one in the Great Caledonian Valley and the other at Glen Treig, and the problem of the Parallel Roads can be solved, provided we allow that glaciers have the power to dam such deep bodies of water as must have occupied Glen Gluoy and Glen Roy."

Mr. Jamieson, in support of this view, adduces the extensive glaciation apparent at the entrance both of Glen Arkaig and of Glen Treig, and shows that near the entrance of Glen Gluoy there are ice striæ, pointing W. 5° N., or in the direction that a glacier coming from Glen Arkaig would take, and that, in the Spean Valley, opposite Glen Treig, the ice striæ are transverse to the valley, or in the direction of the axis of Loch Treig, while on either side they point respectively up and down Glen Spean. He infers, consequently, that the central